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A Spin Glass with Random Couplings

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We define and study a spin glass model based on a RG analysis of its random couplings. The Edwards–Anderson parameter shows a transition.

KEY WORDS: Spin glasses; random coupling; Edward–Anderson transition.

1. INTRODUCTION

Spin-glass models still present something of a mystery: It is not exactly known what they should describe, and therefore it is not obvious what makes a spin-glass model "realistic" or "relevant." Certainly the review by Toulouse[(1)] vividly describes the ambiguous situation in the field. We therefore feel free to add, or rather reemphasize, a commonly somewhat neglected aspect of the question: Namely, we want to view the spin-glass problem as a problem of the *random variables* describing the random couplings. In particular, we are interested in the behavior of the effective random coupling under a *change of scale*. This will lead us naturally to a renormalization group (RG) approach.

This description will become exact in the hierarchical approximation described below (see also Refs. 2 and 3), and we shall describe and study some aspects of the corresponding models which are random version of a Migdal–Kadanoff type of recursion relation.[(4,5)] Alternatively our approach leads us to a study of nonindependent (but *not* strongly coupled (mean-field[(6)])) random variables, and our results can be viewed as an example of nontrivial behavior in this field of mathematics.

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[4] With great sadness we have to inform the reader that Vladimir Jurko Glaser died on January 22, 1984.

The purpose of our paper is to describe and analyze a class of such models, and in particular to study the "evolution" of the effective random coupling as a function of the size of the lattice (Sections 1, 2, and 3).

In Section 4, we shall then arrive at the proof of *existence of a spin-glass transition* in the following sense. For a class of random interactions, we show that at high temperature the expected value of the spin is

$$E(\langle s\rangle) = 0$$

and

$$E(\langle s\rangle^2) = 0$$

where $\langle\ \rangle$ denotes the partition sum and $E(\cdot)$ denotes averaging over the sample space of random couplings. At low temperature, we have

$$E(\langle s\rangle) = 0$$

but

$$E(\langle s\rangle^2) \neq 0$$

indicating a transition of the Edwards–Anderson parameter. We are unable to locate a critical surface, although we can exhibit a critical fixed point.

In order to study this question, we consider a simplifed model in a separate publication[7] in which the probability density ρ of the random coupling is replaced by a discrete sequence

$$\rho_n \sim \int_{1/2^{n+1}}^{1/2^n} (\rho(x) + \rho(-x))\, dx$$

and the renormalization transformation is replaced by the simpler, but—as we believe—qualitatively correct operator defining ρ_n' from ρ_n by

$$\rho_n' = \sum_{p+q=n+1} \rho_p \rho_q + \delta_{n0}\rho_0^2$$

This operator, $T\{\rho_n\} = \{\rho_n'\}$, has two fixed points

$$\{\rho_n\} = \{1, 0, 0, \ldots, \} \equiv S_L \qquad \text{(low temperature)}$$

$$\{\rho_n\} = \{0, 1, 0, \ldots, \} \equiv S_C \qquad \text{(critical temperature)}$$

and a "stable" limit

$$\{\rho_n\} = \{0, 0, \ldots, 0, 1\} \equiv S_H \qquad (\text{i.e., } \rho_\infty = 1) \quad \text{(high temperature)}$$

We show among other things that only three fates can happen to a sequence under iteration by T. If it is S_C it is a fixed point. Otherwise the sequence tends to S_L or to S_H, and, most interestingly, there is a critical surface $\mathscr{H}$, and *every* sequence on it (except S_C) tends to S_L. More precisely, define $f(s) = \sum_{n=0} \rho_n s^n$. The action of T on these generating functions is $(Tf)(s) = (f(s)^2 - f(0)^2)/s + f(0)^2$. Then the critical surface is given by the equation $f(2) - 2f'(2) = 0$, while the high-temperature (h.t.) region is $f(2) - 2f'(2) < 0$, and the low-temperature region is $f(2) - 2f'(2) > 0$. Note that the speed with which the critical surface is left on the h.t. side depends on the initial distribution and thus no critical indices are really defined. This may account for some of the difficulties encountered in spin glasses.

2. THE MODELS

The models we are considering have two equivalent formulations: One is based on a construction of Migdal–Kadanoff,[4,5] while the other is based on the recursive diamond shaped lattices (see Ref. 2).

In the first case, we consider a $\mathbb{Z}^d$ lattice with an Ising spin at each site. In this lattice, we single out a direction, for example the first coordinate $\mathbf{e} = (1, 0, \ldots, 0)$, and we assume that the system has side 2^N in the $\mathbf{e}$ direction and $2^N - 1$ in the others. The following description should be easier to understand by referring to Fig. 1.

To each (horizontal) link of the form $\mathbf{i}, \mathbf{i} + \mathbf{e}$, $\mathbf{i} \in \mathbb{Z}^d$ there is a random coupling $\xi_\mathbf{i}$. Below, we shall specify the nature, independence, ... of the random variables $\xi_\mathbf{i}$. There are now two types of interaction between the Ising spins.

(i) The interaction energy between $s_\mathbf{i}$ and $s_{\mathbf{i}+\mathbf{e}}$ is $\xi_\mathbf{i}$.

(ii) Every hyper-"plane" with fixed first coordinates $i_1 > 0$ is partitioned into $2^{(N+1-r)(d-1)}$ hypercubes of dimension $d-1$ and of side $2^r - 1$, where r is given by

$$i_1 = 2^r + 2^{r_2} + \cdots + 2^{r_k}, \qquad r < r_2 < r_3 < \cdots < r_k$$

$i_1 = 0$ is handled as $i_1 = 2^N$. There is an infinite ferromagnetic coupling in each hypercube, i.e., all spins in one such hypercube are equal.

For a fixed choice of the random variables $\xi = \{\xi_\mathbf{i}\}$, we denote by $H_N(\mathbf{s}, \xi)$ the energy of the spin configuration $\mathbf{s}$. The Gibbs density at inverse temperature β is given by

$$G_n(\mathbf{s}, \xi, \beta) = \frac{\exp[-\beta H_N(\mathbf{s}, \xi)]}{\sum_{\mathbf{s}'} \exp[-\beta H_N(\mathbf{s}', \xi)]}$$

We are interested in the properties of G_N as a function of β in the thermodynamic limit, $N \to \infty$.

We now give a second description of the model, using a recursive buildup of a (hierarchical) lattice. This formulation has a natural extension

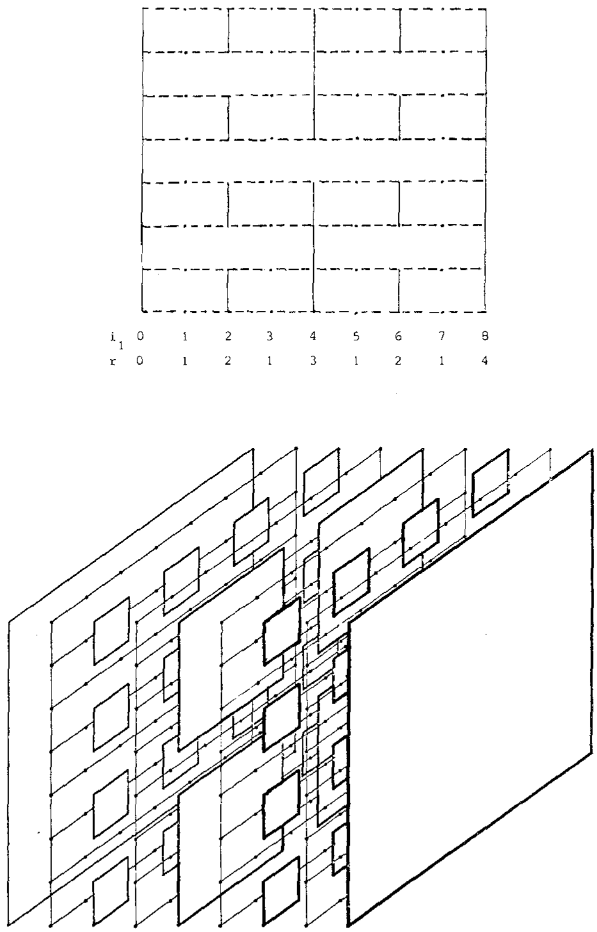


Fig. 1. (a) Lattice for $d = 2$, $N = 3$, **e** horizontal. (b) Lattice for $d = 3$, $N = 3$, **e** horizontal.

to noninteger dimensions, as we shall see. One first chooses an integer $n \geqslant 1$ (one should think of $n = 2^{d-1}$, in the first formulation of the model). The lattice is then formed recursively as follows. The first lattice is formed by two sites, and one link (see Fig. 2).

We call this L_0. If L_p, $p \geqslant 0$, has been constructed, then L_{p+1} is obtained by replacing each link by n sites and $2n$ links connecting each new site to the two ends of the original link (see Fig. 3).

We now consider L_N. To each site i (numbered in some suitable fashion) we associate an Ising spin, and to each link (i, j) a random coupling constant $\xi_{i,j}$. Again, for each fixed choice of the random variables,

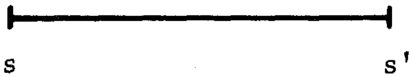


Fig. 2. One bond.

we can define

$$H_N(\mathbf{s}, \xi) = \sum_{\substack{i,j \\ \text{(nearest neighbors)}}} \xi_{i,j} s_i s_j$$

The Gibbs measure is defined as before.

So far, we have said nothing about the nature of the random variables, and the model still leaves us some freedom of choice. The most interesting choice is that of independent (identically distributed) random variables. We discuss this case in a separate publication.[8] The main difference of that variant as contrasted to the one presented below is the presence of frustration. But it should be stressed that even in the simplified model of this paper, the absence of frustration is not the same as talking about a purely ferromagnetic interaction, as we shall see. Alternately, our model can be viewed as having *random* ferromagnetic interactions.

The restriction we are going to make in this paper is the following. In the Migdal–Kadanoff version, the restriction is

$$\xi_{\mathbf{i}} = \xi_{i_1}$$

i.e., all random variables whose index $\mathbf{i}$ has the same first component are equal. It is easy to give a similar formulation of this condition in the diamond version of the model; the details are left to the reader.

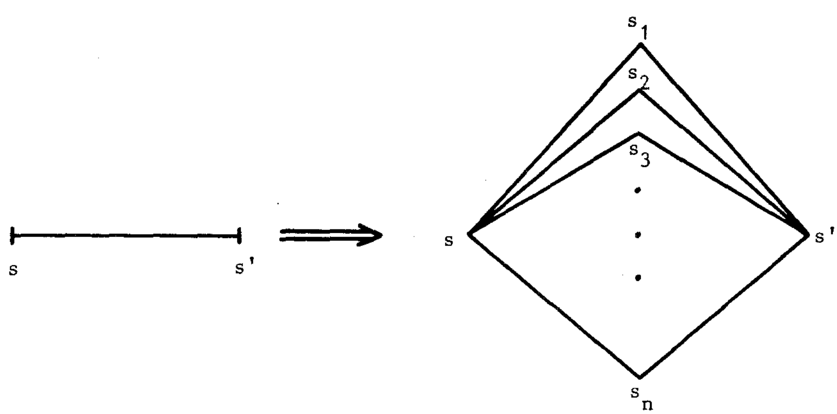


Fig. 3. Increasing the level by one.

The point of the above models is that the Migdal renormalization transformation[5] is exact.

3. RENORMALIZATION TRANSFORMATION

In this section, we establish Migdal's recursion relations, which are exact for the models we have described. Consider L_N. The renormalization consists in summing over all spins introduced in the step leading from L_{N-1} to L_N. The resulting lattice L_{N-1} will have new effective (random) coupling constants. We now derive the formula for getting the new couplings $\hat{\xi}$ as a function of the old ones. It is clearly sufficient to consider the following situation (see Fig. 4).

We reemphasize that we want to sum over $s_1, \ldots, s_n$ and replace $\xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_n$ by a new random variable $\hat{\xi}$ giving an effective coupling between s and s'.

It is useful to introduce the random variables $x_i = \tanh(\xi_i)$, $x'_i = \tanh(\xi'_i)$. Moreover, we shall assume that the inverse temperature has been absorbed in the definition of the ξ_i. We then have to compute

$$I = \sum_{\substack{s_1, \ldots, s_n \\ = \pm 1}} \prod_{i=1}^{n} \left(e^{\xi_i s s_i} e^{\xi'_i s_i s'}\right)$$

$$= \sum_{\substack{s_1, \ldots, s_n \\ = \pm 1}} \prod_{i=1}^{n} \cosh \xi_i \cdot \cosh \xi'_i \cdot (1 + x_i s s_i)(1 + x'_i s_i s')$$

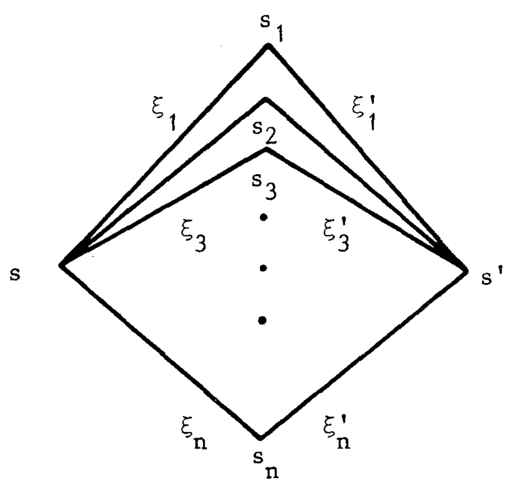


Fig. 4. Labeling one diamond.

The factors $\cosh \xi_i \cdot \cosh \xi_i'$ will eventually disappear in the normalization of the partition function. We shall omit them henceforth. The quantity to study is thus

$$\prod_{i=1}^{n} \sum_{s_i=\pm 1} (1 + x_i s s_i)(1 + x_i' s_i s')$$

$$= 2^n \prod_{i=1}^{n} \frac{1}{\cosh\left[\tanh^{-1}(x_i x_i')\right]}$$

$$\times \prod_{i=1}^{n} \left\{\cosh\left[\tanh^{-1}(x_i x_i')\right] + \sinh\left[\tanh^{-1}(x_i x_i')\right] s s'\right\}$$

$$= 2^n \prod_{i=1}^{n} \frac{1}{\cosh\left[\tanh^{-1}(x_i x_i')\right]} \exp\left[\sum_{i=1}^{n} \tanh^{-1}(x_i x_i') s s'\right]$$

$$= 2^n \cosh\left[\sum_{i=1}^{n} \tanh^{-1}(x_i x_i')\right] \frac{1 + \tanh\left[\sum_{i=1}^{n} \tanh^{-1}(x_i x_i')\right] s s'}{\prod_{i=1}^{n} \cosh\left[\tanh^{-1}(x_i x_i')\right]}$$

We again omit the factors which do not depend on s, s' (and which disappear in the normalization), and we get the transformation for the ξ:

$$\zeta = \sum_{i=1}^{n} \tanh^{-1}\left[\tanh(\xi_i)\tanh(\xi_i')\right]$$

In the case of interest for this paper, the ξ_i are equal and so are the ξ_i' [say, to (ξ, ξ')] and we get

$$\zeta = n \tanh^{-1}\left[\tanh(\xi)\tanh(\xi')\right]$$

or, in terms of the x,

$$z = \tanh\left[n \,\mathrm{tahn}^{-1}(x x')\right]$$

It is useful to denote

$$q_n(s) = \tanh\left[n \tanh^{-1}(s)\right]$$

If all the x_i have probability density f, then the probability density of the renormalized coupling z will be

$$\frac{1}{|q_n'(q_n^{-1}(t))|} \int_{-1}^{1} f(x) f\left(\frac{q_n^{-1}(t)}{x}\right) \frac{dx}{|x|} \tag{3.1}$$

This transformation is the main object of study of our paper.

For completeness, we also give the renormalization of a correlation function, for one spin. This quantity is given by

$$\mathscr{J}(s_1) = \frac{1}{I} \sum_{s_1, \ldots, s_n} s_1 \prod_{i=1}^{n} \cosh(\xi_i)\cosh(\xi_i')(1 + x_i s s_i)(1 + x_i' s_i s')$$

$$= \frac{x_1(1 - x_1'^2)s + x_1'(1 - x_1^2)s'}{1 - x_1^2 x_1'^2} \cdot \left\{ 1 + \tanh\left[\sum_{i=1}^{n} \tanh^{-1}(x_i x_i') s s' \right] \right\} \tag{3.2}$$

The identities (3.1) and (3.2) will allow us to compute explicitly the models L_N by N-fold iteration, and we shall be able to take the thermodynamic limit.

4. BEHAVIOR UNDER RENORMALIZATION

In this section, we investigate the action of the renormalization transformation on the probability law (i.e., the probability density).

It is easy to verify that the transformation (3.1) has three fixed probability laws, given by

$$\delta_0$$

$$(\delta_1 + \delta_{-1})/2$$

and

$$(\delta_{a_n} + \delta_{-a_n})/2, \qquad \text{where} \quad q_n(a_n^2) = a_n$$

We do not know whether there are other fixed points, but we conjecture that there are none. We now prove some results about the basin of attraction of these three fixed points. Our analysis is not as complete as one could wish, but in the simplified case of[7] we shall be able to get a picture which is probably the same as the one to be expected for the transformation (3.1).

We shall denote by x_m the random coupling obtained after m steps of renormalization from x_0. If x_m' denotes an independent identical copy of x_m, then x_{m+1} is in fact nothing else than

$$x_{m+1} = q_n(x_m \cdot x_m').$$

We shall now study the limiting behavior of this recursion, and for this purpose, we state some preliminary estimates. Henceforth we shall fix n, and denote $q = q_n$, $a = a_n$. We also define

$$P_m(x) = \mathrm{Prob}\{|x_m| \leqslant x\}$$

Lemma 4.1. The following inequalities hold:

(Ei) One has $P_{m+1}(x) \geqslant P_m(h(x))^2$, where $h(x) = [q^{-1}(x)]^{1/2}$.

(Eii) If $P_m(y) = P_m(a)$ for $y \geqslant a$ then $0 \leqslant x \leqslant a$, implies $P_{m+1}(x) \geqslant 2P_m(g(x)) - P_m(g(x))^2$, where $g(x) = q^{-1}(x)/a$.

(Eiii) If $P_m(y) = 0$ for $y < a$, then, for $a \leqslant x \leqslant 1$, $P_{m+1}(x) \leqslant P_m(g(x))^2$.

(Eiv) If $P_m(y) = 0$ for $y < a$, then, for $a \leqslant x \leqslant 1$, $P_{m+1}(x) \leqslant 2P_m(h(x))$.

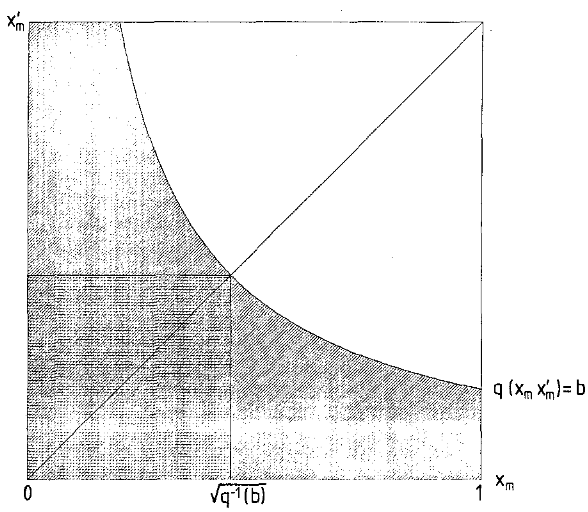
(Ev) $P_{m+1}(x) \geqslant 2P_m(q^{-1}(x)) - P_m(q^{-1}(x))^2$.

Proof. If we view $P_{m+1}(x)$ as a double integration, over $q(x_m x'_m) \leqslant x$, then the five inequalities are straightforward consequences of restrictions of these domains of integration. We visualize them graphically (see Fig. 5). ■

We can now use these estimates to prove convergence. We shall consider only even distributions.

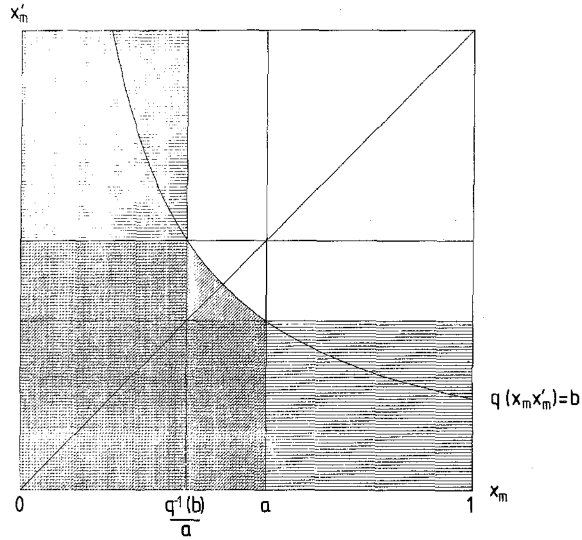
Lemma 4.2. If $x_0 \in [-a, a]$, almost surely, and $P_0(a - \epsilon) > 0$ for some $\epsilon > 0$, then, almost surely $x_m \to 0$ as $m \to \infty$.

Proof. We have $P_0(y) = P_0(a)$ for $y \geqslant a$ by assumption, and since a is a repelling fixed point for $q(z^2)$, this implies $P_m(y) = P_m(a)$ for all m, and $y \geqslant a$. Thus (Ei), (Eii) apply.

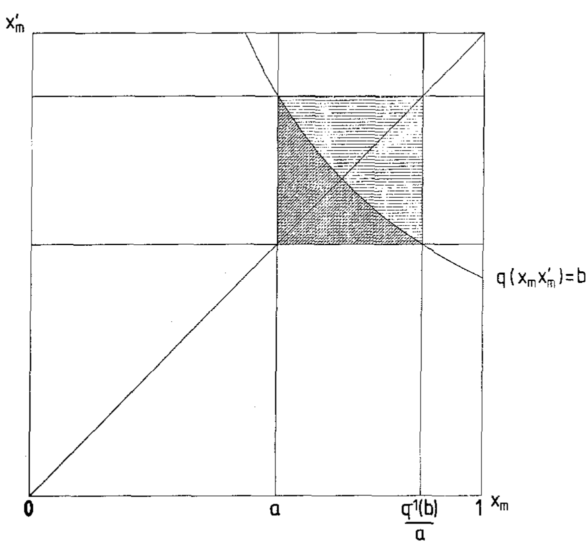


(a)

Fig. 5. Integration regions.



(b)



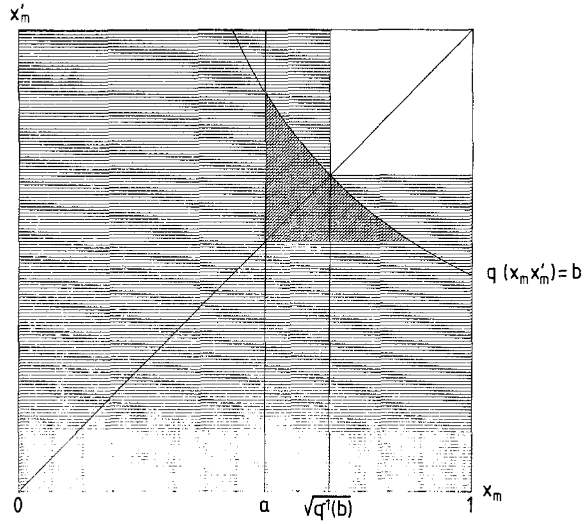
(c)

Fig. 5. Continued.

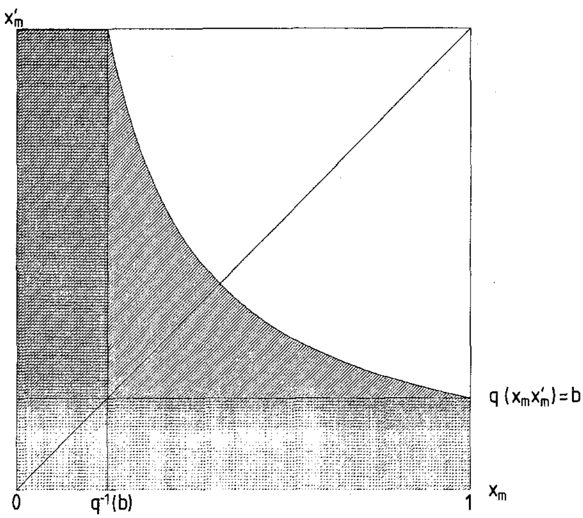
For $x \leqslant a$, we have

$$P_{m+2}(x) = k(P_m(h \circ g(x)))$$

where $k(z) = 2z^2 - z^4$. The map $z \to k(z)$ has a stable fixed point at 1, which attracts the interval $(\sigma, 1]$ where $\sigma = (\sqrt{5} - 1)/2$. Note also that $g^{-1} \circ h^{-1}$ has a stable fixed point at 0 which attracts $[0, a)$. Therefore, for



(d)



(e)

Fig. 5. Continued.

$x \in [0, a)$ we find

$$P_{m+2p}(x) \geqslant P_{m+2p}\big(g^{-1} \circ h^{-1}(x)\big) \geqslant k\big(P_{m+2(p-1)}(x)\big)$$
$$\geqslant k^p\big(P_m(x)\big)$$

Thus if $P_m(x) > \sigma$, then $P_{m+2p}(x) \to 1$ as $p \to \infty$.

Consider now $l(x) = 2x - x^2$. The map $z \to l(z)$ has a stable fixed point at 1, which attracts (0, 1]. Since $P_0(a) \neq 1$, and $|x_0| \leqslant a$ almost surely there is a $y < a$ for which $P_m(y) \neq 0$. Therefore, there is an s for which $l^s(y) > \sigma$, and hence, by (Eii), for $x = g^{-s}(y)$,

$$P_s(x) \geqslant l^s(P_0(y)) > \sigma$$

and hence,

$$P_{s+2p}(x) \to 1$$

Since P_m is monotone, and $g(x) < x$ when $x < a$, we have

$$P_{s+2p}\left(g^{-1}(x)\right) \to 1$$

But this implies, by (Eii),

$$P_{s+2p-1}(x)^2 \to 1$$

and the lemma follows. ■

It is interesting to note that $P_p(y)$ in general will not converge monotonously to 1, and in fact the nature of the convergence will depend very much on the density of x_0. We are confronted with two opposing tendencies: Lemma 4.1 tells us that the weight of the density moves toward 0, but it also gets smaller. All these facts will become more transparent in the simplified model of Ref. 7. They are responsible for the absence of critical indices at the critical surface. The following lemma is a sort of converse of Lemma 4.2.

Lemma 4.3. Assume $|x_0| \geqslant a$ almost surely and $P_0(a) \neq 1$. Then $|x_m| \to 1$ almost surely, as $m \to \infty$.

Proof. The construction we give now has to go somewhat backwards. Let $\alpha = P_0(a)$. By assumption, we have $\alpha < 1$. We denote $\alpha' = (1 + \alpha)/2$ (< 1). Define s by

$$(\alpha')^{2^s} < 2^{-(r+1)}$$

Choose $b > a$ such that (i) $q^{-1}(b) < a$, (ii) $P_0(g^s(b)) \leqslant \alpha'$, (iii) $g^s(b) < c$, where c is the boundary of the basin of attraction of $h^r \circ g$. Such a b exists (it suffices to choose it sufficiently close to a).

Assume now $d \in [a, 1)$, and choose q such that $h^q(d) \leqslant b$. Then we find, using (Eiv),

$$P_{q+m}(d) \leqslant 2^q P_m(h^q(d)) \leqslant 2^q P_m(b)$$

It suffices therefore to prove that $P_m(b) \to 0$ as $m \to \infty$. By (ii) above

$$P_s(b) \leqslant P_0(g^s(b))^{2^s} \leqslant \alpha'^{2^s} < 2^{-(r+1)}$$

Applying alternately the two inequalities (Eiii) and (Eiv) we see that

$$\begin{aligned} P_{s+p(r+1)}(b) &\leqslant P_{p(r+1)}(g^s(b))^{2^s} \\ &\leqslant 2^r P_{(p-1)(r+1)}(h^r \circ g)((g^s(b)))^{2^s \cdot 2} \\ &\leqslant 2^r P_{(p-1)(r+1)}(g^s(b))^{2^s \cdot 2} \end{aligned}$$

since (iii) implies $(h^r \circ g)(g^s(b)) < g^s(b)$. Iterating, we get

$$P_{s+p(r+1)}(b) \leqslant 2^{r(2^p-1)}\left(P_0(g^s(b))^{2^s}\right)^{2^p} \leqslant 2^{-2^p} \to 0$$

by (Eii) and (Eiii). Since $h(x) < x$, for $x > a$, and P_m is monotone, we have

$$\begin{aligned} P_{s+p(r+1)+q}(b) &\leqslant P_{s+p(r+1)+q}\left(h^{-(r-q+1)}(b)\right) \\ &\leqslant 2^q P_{m+p(r+1)}(b) \to 0, \qquad \text{for} \quad q = 0, \ldots, r \end{aligned}$$

This completes the proof. ■

We can now analyze easily the temperature dependence for a subclass of random variables ξ_0 and show that there is a phase transition, as far as the random couplings are concerned. We postpone to the next section the physical aspects of this transition.

Theorem 4.4. Let ξ_0 be a random variable such that for some γ_1, γ_2,

$$0 < \gamma_1 \leqslant |\xi_0| \leqslant \gamma_2 < 1, \qquad \text{almost surely (a.s.)}$$

If the temperature T is sufficiently large, the associated sequence x_m tends to zero, almost surely, while for T sufficiently small, $|x_m|$ tends to 1 almost surely.

Proof. By construction, we have $x_0 = \tanh(\beta\xi_0)$ with $\beta = 1/T$, and therefore

$$\tanh(\beta\gamma_1) \leqslant |x_0| \leqslant \tanh(\beta\gamma_2), \qquad \text{a.s.}$$

Therefore, by Lemma 4.2, if $\beta < \gamma_2^{-1}\tanh^{-1}(a)$, we have $x_m \to 0$ a.s., while, by Lemma 4.3, if $\beta > \gamma_1^{-1}\tanh^{-1}(a)$, then $|x_m| \to 1$, a.s. ■

When ξ_0 is of the type described in Theorem 4.4, one could be tempted to identify the critical temperature as the largest temperature for which $|\tanh(\beta\xi_0)| \leqslant a$, a.s. This is, however, not the case, as can be shown by an explicit example, which we do not present here. [If $|\xi_0|$ is a.s. constant, the critical temperature *is* characterized by $|\tanh(\beta\xi_0)| = a$, a.s.]

We next discuss a more detailed issue—a description of the basin of attraction of $x = 0$. Our description is not complete, but it shows that the domain of attraction has a structure which is not very simple in the L_1 topology. This is due to the nondifferentiability of the renormalization map in this space.

We present here the result only for the case $n = 2$, i.e., $q = q_2$. Other cases are similar, and better constants can be read off the proof.

Lemma 4.5. If, for some $0 < \alpha \leqslant 1/128$, one has $P_0(\alpha) > 4\alpha$, then $x_m \to 0$ almost surely.

Proof. By the inequalities (Ei), (Ev), we have

$$P_{p+1+m}(x) \geqslant v \circ u^p(P_m(q^{-p} \circ h(x)))$$

where $v(z) = z^2$, $u(z) = 2z - z^2$. Below, we shall show that

$$v \circ u^p \text{ attracts } (\tau_p/4^p, 1] \text{ to } 1 \tag{4.1}$$

and $\tau_p \to 1$ as $p \to \infty$ ($\tau_3 \sim 1.45$).

On the other hand, since $q(x) = q_2(x) = 2x/(1 + x^2)$, we have $q^{-1}(x) > x/2$ and $h(x) > (x/2)^{1/2}$. Therefore, if $x = 1/2^{2p+1}$, then $q^{-p} \circ h(x) > x$. Assume now

$$P_0(1/2^{2p+1}) > 2\tau_p/4^p$$

Then, setting $x = 1/2^{2p+1}$, we have

$$P_{ps}(x) \geqslant v \circ u^p(P_{p(s-1)}(q^{-p} \circ h(x)))$$
$$\geqslant v \circ u^p(P_{p(s-1)}(x)) \geqslant (v \circ u^p)^s(2\tau_p/4^p) \to 1$$

This implies by (Ei),

$$P_{ps+1}(q(w^2)) \geqslant P_{ps}(w) \to 1$$

But $q(1/2^{2p+1})^2 < 2(2^{2p+1})^2$, and repeating this argument, we see that $P_m(x) \to 1$ for all $x > 0$. It remains to prove (4.1). This is an easy consequence of the inequality, obtained by induction,

$$u^p(z) \geqslant 2^p z - 2^{p-1}(2^p - 1)z^2$$

and of

$$v \circ u^p(z) \geqslant 4^p z^2 - 4^p(2^p - 1)z^3$$

5. SPIN OBSERVABLES

In this section, we investigate the behavior of the expectation of the spin when the "volume" of the lattice tends to infinity. We fix the values of the spin at the two extreme points of the lattice, thereby choosing the boundary conditions. If s is a spin which is not one of the above two, we shall denote by $\langle s \rangle$ its expectation for fixed boundary condition, and fixed values of the random couplings. We show that the expectation $E(\langle s \rangle^2)$ (i.e., average over couplings) of $\langle s\rangle^2$ satisfies

$$E(\langle s\rangle^2)\begin{cases} = 0, & \text{at high temperature} \\ \neq 0, & \text{at low temperature} \end{cases}$$

and

$$E(\langle s\rangle) = 0 \qquad \text{at all temperatures}$$

Note that the above argument shows a transition for the Edwards–Anderson[1,3] parameter. We shall also see that the value of the Edwards–Anderson parameter is independent of the boundary conditions.

Consider the lattice L_N and a fixed spin variable s_0 which has been "created" at level N. The variable s_0 has two neighboring sites, exactly one of which has been created at level $N-1$. We call it s_1 and we call the other neighbor s'_1. Considering now s_1 as a fixed spin in L_{N-1}, we find its neighbors s_2 and s'_2 in the same fashion as before, and continuing inductively, we find two chains $s_0, s_1, \ldots, s_N$ and $s_0, s'_1, s'_2, \ldots, s'_N$ of spins. Note that all s_i are distinct, but some of the s'_i may coincide. (See Fig. 6.) Note also that either $s_{m+1} = s'_m$ or $s'_{m+1} = s'_m$.

To these chains of spins, we associate functions of the form

$$F_i(s_i, s'_i) = a_i s_i + b_i s'_i$$

where the a_i, b_i are real functions depending on the couplings x_0. We are interested in $\langle F_i(s_i, s'_i)\rangle$ [i.e., the canonical expectation of the observable $F_i(s_i, s'_i)$]. We have the following important identity:

$$\langle As_m + Bs'_m\rangle = \langle A's_{m+1} + B's'_{m+1}\rangle$$

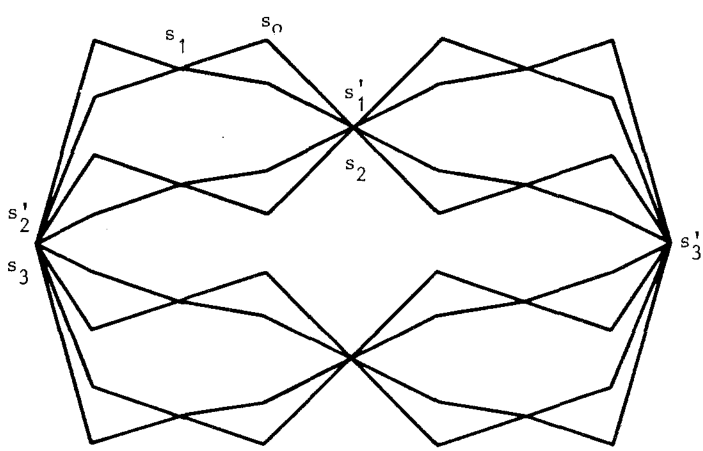


Fig. 6. Labeling of the spins.

where

$$A' = B + A\frac{x_m(1 - x_m'^2)}{1 - x_m^2 x_m'^2} \qquad \text{if} \quad s_{m+1} = s_m' \tag{5.1}$$
$$B' = A\frac{x_m'(1 - x_m^2)}{1 - x_m^2 x_m'^2}$$

and

$$A' = A\frac{x_m(1 - x_m'^2)}{1 - x_m^2 x_m'^2} \qquad \text{if} \quad s_{m+1}' = s_m' \tag{5.2}$$
$$B' = B + A\frac{x_m'(1 - x_m^2)}{1 - x_m^2 x_m'^2}$$

Here, x_m and x_m' are two identical, independent random couplings obtained from x_0 through m-fold application of the RG transformation, $x_m = \tanh(T^m\xi)$. The above identities are immediately obtained by induction when summing over all spins at level $N - m$.

Namely,

$$(1 + x_m s_m s_{m+1})(1 + x_m' s_m s_{m+1}') = (1 + x_m x_m' s_{m+1} s_{m+1}')$$
$$\times \left(1 + \frac{x_m s_{m+1} + x_m' s_{m+1}'}{1 + x_m x_m' s_{m+1} s_{m+1}'} s_m\right)$$

and, summing over s_m (and taking the mean, we get)

$$\sum_{s_m}\left(1 + \frac{x_m s_{m+1} + x_m' s_{m+1}'}{1 + x_m x_m' s_{m+1} s_{m+1}'} s_m\right) \cdot (A s_m + B s_m')$$
$$= A\left(\frac{x_m s_{m+1} + x_m' s_{m+1}'}{1 + x_m x_m' s_{m+1} s_{m+1}'}\right) + B s_m'$$

The equations follow from $s_m' = s_{m+1}$ (resp. $= s_{m+1}'$).

Given s_0 and $A_0 \neq 0$, we therefore find, when $B_0 = 0$,

$$\langle A_0 s_0\rangle = \langle A_1 s_1 + B_1 s_1'\rangle$$
$$= \cdots$$
$$= A_n s_N + B_N s_N' \tag{5.3}$$

where the A_i, B_i are recursively obtained by the above relations, and are random variables in all $x_j, x_j' = \tanh(\xi_j), \tanh(\xi_j')$ created above level i.

Lemma 5.1. $E(A_m B_m) = E(A_m) = E(B_m) = 0.$

Proof. We shall show below, recursively, that

$$E\big(A_m x_m f(x_m^2)\big) = E\big(B_m x_m f(x_m^2)\big) = 0 \tag{5.4}$$

for any function f.

Using this result, and the induction rule for forming A_{m+1} from A_m, B_m, we get, in case $s_{m+1} = s'_m$

$$E(A_{m+1}) = E(A_m) + E\left(B_m x_m \frac{1 - x_m^2}{1 - x_m^2 x_m'^2} \right)$$

from which the assertion $E(A_{m+1}) = 0$ follows. Similarly, $E(B_{m+1}) = 0$. Finally,

$$E(A_{m+1}B_{m+1}) = E\left(A_m B_m x'_m \frac{1 - x_m^2}{1 - x_m^2 x_m'^2} \right)$$

$$+ E\left[B_m^2 \left[\frac{x_m x'_m (1 - x_m^2)(1 - x_m'^2)}{(1 - x_m'^2 x_m^2)^2} \right]\right] = 0$$

by (5.1). It remains to prove (5.4). But this is obvious since

$$E\big(A_m x_m f(x_m^2)\big) = E\big(A_{m-1} x_{m-1} \Phi(x_{m-1}^2, x_{m-1}'^2)\big)$$

where

$$\Phi(x_{m-1}^2, x_{m-1}'^2) = x_{m-1}^2 (1 - x_{m-1}'^2) f\big(q_n(x_{m-1} x'_{m-1})^2\big)$$

and so the result follows by induction. ■

We next show recursively that

$$A_{m+1} B_{m+1} x_{m+1} \geqslant 0 \qquad \text{a.s.}$$

Indeed, this follows again by induction, if we use

$$A_{m+1} B_{m+1} x_{m+1} = \frac{x_m A_m B_m x'_m}{1 - x_m^2 x_m'^2} \frac{(1 - x_m^2) x_m}{x_m^2} q_n(x_m x'_m)$$

$$+ B_m^2 \frac{(1 - x_m^2)(1 - x_m'^2)}{x_m^2} q_n(x_m x'_m)$$

$$\geqslant 0$$

since $x_m x'_m q_n(x_m x'_m) \geqslant 0$ and $A_m B_m x_m \geqslant 0$, a.s.

Therefore,

$$|A_{m+1}| + |B_{m+1}| = |A_m| + |B_m| \frac{|x_m| + |x'_m|}{1 + |x_m x'_m|} \tag{5.5}$$

as is easily seen.

Assume now $|x_m| \to 1$, exponentially fast (this certainly happens for the x_0 of Theorem 4.4).
Then

$$|A_m| + |B_m| \geqslant (|A_0| + |B_0|)\text{const},$$

by (5.5) and this (positive) constant does not depend on m. If we set $A_0 = 1$, $B_0 = 0$, and we now take $m = N$, we obtain

$$\langle s_0 \rangle = \langle A_N s_N + B_N s'_N \rangle = A_N s_N + B_N s'_N$$

From Lemma 5.1 this implies $E(\langle s_0 \rangle) = 0$. On the other hand,

$$E(\langle s_0 \rangle^2) = E(A_n^2 + B_n^2) + E(A_N B_N) s_N s'_N$$

$$= E(A_n^2 + B_N^2) \geqslant \tfrac{1}{2} E((|A_N| + |B_N|)^2) > 0$$

Hence we have shown for any fixed s_N, s'_N the following:

Theorem 5.2. If $|x_m| \to 1$ or 0 a.s. exponentially fast then

$$E(\langle s \rangle) = 0$$

$E(\langle s \rangle^2)$ has a limit which is not zero if $|x_m| \to 1$ and zero otherwise.

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